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## THE CONICS OF APOLLONIUS.

APOLLONIUS was born at Perga in Pamphylia, in the south of Asia Minor. The date usually assigned to him is 240 B.C., so that he was about seventy years later than Euclid and about thirty years later than Archimedes. He tells us that it was at the suggestion of the geometer Naucrates that he wrote his treatise on Conics, consisting of eight books, and that he composed them somewhat hurriedly, and submitted them to him without revision, simply putting down the different properties as they occurred to him, intending to go over them again afterwards, the reason for his haste being the approaching departure of his friend and guest. He afterwards found time to revise and publish each book separately, and it is in this revised condition that they have come down to us. His own general summary of their contents is as follows:—

"The first four books deal with the elements of the subject. The first book treats of the generation of the three sections of a cone, and of the two branches of a hyperbola, and their principal properties are more fully and generally developed than is the case in the writings of others. The second contains properties relating to the diameters and axes of conics, and asymptotes, and other matters which are indispensable for distinguishing between different cases of propositions. The third book contains many curious theorems which are useful for the solution and discrimination between the different cases of solid problems, of which theorems the greater number and the most interesting are new; by means of which I was able to investigate thoroughly the *locus ad tres et quatuor lineas*, which was incompletely solved by Euclid, for that which he chanced to hit upon was only a particular case, and this he solved in a manner that was not satisfactory, for it was not possible for the solution to be completed without the aid of the properties which I have discovered. The fourth book shows in how many ways conics intersect one another, and the circumference of a circle, and other out-of-the-way properties, none of which have been investigated by previous writers, such as in how many points the section of a cone, or the circumference of a circle, or both branches of a hyperbola, intersect the two branches of another hyperbola. The other books are more advanced. One deals fully with questions of maxima and minima, a second with the equality and similarity of sections, a third with theorems which separate conics into their different kinds, and the last with problems on the different kinds of conics."

Of these eight books we possess the first four in Greek, and we know that the last four were extant in that language in the time of Pappus in the fourth century, and later still in the time of Eutocius in the sixth century; but after this all trace of them disappears until Books V., VI., VII. are met with in an Arabic translation made by Thebit ben Corah, about 800 A.D., and afterwards revised by Nasir Eddîn, the Persian mathematician, about 1250 A.D. In addition to this there is an epitome in Arabic of the same three books made by Abdolmelec, a Persian, about 820 A.D., and another made by Abalphath of Ispahan. The latter was translated into Latin by Abraham Ecchellensis, Professor of Oriental Languages at Florence. As he had no knowledge of mathematics he called in to his assistance Alphonsus Borellus, Professor of Mathematics at Pisa, who was unacquainted with Arabic. To add to their difficulties the diacritical points were wanting, so that the letters were, as it were, mere matter without form. However, between them they managed in three months to complete their version, which, with the addition of copious notes by Borellus, was published at Florence in 1661.

In 1710 Edmund Halley, the Savilian Professor of Geometry, published at Oxford the first four books in Greek and Latin, and a Latin translation of Thebit ben Corah's Arabic version. Of the eighth book, which Apollonius tells us was a sort of appendix to the seventh, we have only Pappus's *Lemmas*, and from them Halley composed a conjectural restitution of it.

In 1891 a volume appeared in the Teubner series containing the first three books in Greek and Latin, edited by Dr. Heiberg, and he promises to bring out another volume containing the fourth book, together with the *Lemmas* of Pappus and the *Commentaries* of Eutocius, but this second volume has not yet appeared.

The work, then, as we possess it, consists of seven books, containing an aggregate of 381 propositions, and I propose in my paper to give a short analysis of the contents of each book.

It should be noticed, in the first place, that the subject is developed not with reference to an axis, but to a diameter, which is defined as being a straight line which bisects a series of parallel chords. The first book contains fifty-six propositions, of which the first ten deal with the sections of a cone made by a plane in the particular cases where the plane passes through the vertex, or is parallel or antiparallel to the base, etc. In Fig. 1 ABC is a scalene cone, which is cut by a plane ABC through the axis, and also by another plane PQP′, of which ZH, the intersection of the plane PQP′ with the plane of the base, is perpendicular to BC. Q is any point on the curve of intersection of the plane PQP′ with the cone, and QVQ′ is drawn in that plane parallel to ZH. Prop. 7 shows that PP′ bisects QQ′, and that the two lines are only perpendicular in the particular cases (1) when the cone is a right cone, and (2) in an oblique cone when the plane through the axis is perpendicular to the base. Hence PP′ fulfils the general definition of a diameter of which QV is an ordinate. In Props. 11, 12, 13, where he discriminates between the three kinds of sections of a cone, he takes the general case of an oblique cone, and consequently the relations are at once established, not between the axis and its ordinates, but between a diameter and its ordinates; and to this diameter, which is the intersection of the cutting plane with the plane through the axis, and which we may call the primary diameter, he confines himself for the present.

The next point that we have to consider is the meaning of the term "latus rectum." With Apollonius the term belonged not so much to a curve as to a diameter, and every diameter had its own latus rectum. For any diameter PP′, in Fig. 1, if AK is parallel to PP′, the latus rectum of PP′ is obtained by drawing PR perpendicular to PP′, and taking

$$PR : PP' = KB \cdot KC : AK^2.$$

In the same figure complete the rectangle RPVF, having the latus rectum PR and the abscissa PV for adjacent sides. Join P′R, meeting FV in E, and draw EO parallel to PV. Then if the cutting plane is not parallel to one of the sides of the cone, Props. 12, 13 show that for any point Q on the section we have the relation

$$QV^2 = PV \cdot PO,$$

or employing the language of Euclid, "The square on the ordinate is equal to a rectangle applied to the latus rectum, having its breadth equal to the abscissa, and exceeding or deficient by a certain other rectangle." And here I would remark that there are two propositions of Euclid with which the student of Apollonius must become familiar, viz. Props. 28 and 29 of the sixth book. Without these he cannot advance a step, as they are constantly being employed or referred to, but he will look in vain for them in the modern editions of Euclid. Most editors are satisfied with giving us the enunciations, with a short reference, in which they condemn them as useless and inelegant, and claim our gratitude for having withheld the demonstrations of them. As a rule they speak of them as being quite useless, or suggest that they have been introduced merely to give a solution of Prop. 30, *i.e.* to cut a given line in extreme and mean ratio. Even de Morgan says that their use is only historical, as "they show how Euclid in the tenth book proceeded in cases of which the arithmetical counterpart would require the solution of quadratic equations."

Returning to Apollonius, we see that the latus rectum of a diameter is a straight line to which is applied a rectangle which is equal to the square on an ordinate with or without the modification described in Euc. VI. 28, 29. The actual nature of the modification gives us the derivation of the names which have been given to the different species of conics. As Dr. Taylor points out in his Geometry of Conics, on p. xxv of the *Prolegomena*, the term παραβολή denotes the construction of a parallelogram on a given line, or as it is sometimes expressed, the *application* of a parallelogram to a given line, as in Euc. I. 44, 45, and this term of application is extended to the cases where the parallelogram is applied either to a part of the given line, or to a line made up of the given line and a part of it produced, the difference between the parallelogram so described and the parallelogram described upon the whole line being similar to another given parallelogram, which are the cases considered in Euc. VI. 28, 29. Now in the case where the square on the ordinate is equal to the parallelogram which is applied to the latus rectum having for its breadth the abscissa, *i.e.* when the application is to the whole of the latus rectum without modification, the section is called a parabola. When the parallelogram is applied to only a part of the latus rectum, so that the square on the ordinate is less than the parallelogram on the whole line, the curve is called an ellipse (from the Greek ἔλλειψις) in consequence of this deficiency. When the application is to a line which exceeds the latus rectum, so that the square on the ordinate exceeds the parallelogram on the latus rectum, the conic is called a hyperbola (from the Greek ὑπερβολή), signifying excess. Having thus defined his curves and obtained a fundamental property connecting the primary diameter with its latus rectum, ordinates, and abscissæ, Apollonius proceeds to find its conjugate, or as he calls it, the secondary diameter.

In the ellipse, if PP′ is the primary diameter, he draws a line DD′ through the centre parallel to the ordinates of PP′, and intersecting the curve in DD′, and it is shown in Prop. 15 that if DR′ is a line perpendicular to DD′ such that

$$DD' : PP' = PP' : DR',$$

DD′ is the diameter conjugate to PP′, and DR′ is the latus rectum of DD′, and the ellipse might be described with reference to DD′ as a primary diameter, so that in the

case of the ellipse he has obtained a pair of conjugate diameters fully determined both as regards direction and magnitude.

In the case of the opposite branches of a hyperbola it is shown in Prop. 16 that if PCP′ is the primary transverse diameter, and a line is drawn through the centre parallel to the ordinates of PP′, it will bisect all chords parallel to PP′, and it will therefore be the direction of the diameter conjugate to PP′. Of course this property merely applies to direction. To obtain the length of the conjugate diameter he gives the definition: "Let the line which is drawn from the centre parallel to the ordinates of the transverse diameter, and which is a mean proportional between the transverse diameter and its latus rectum and is bisected by the centre, be called a secondary diameter." Thus in the hyperbola as well as in the ellipse the primary diameter and its conjugate are completely determined not only in direction but also in magnitude, the latter by its intersection with the curve, and the former by the definition that it is the mean proportional between two known straight lines.

He then leaves the cone and investigates the properties of conics as being curves in a plane in which the above relations hold between the ordinates, abscissæ, and latus rectum of a diameter. At this point I would notice his method of treating the different sections of a cone. If there is any property which is common to two or more of them, he states definitely to which of them it applies. If one or more of them require special treatment, he proves the property for those cases separately. When it is necessary to do this, the order observed is, (1) the parabola, (2) one branch of the hyperbola, (3) the ellipse, (4) both branches of the hyperbola. In using the term hyperbola the meaning is always confined to one branch of the curve. When both branches are referred to, they are spoken of as "the opposite sections," and a separate demonstration is considered to be necessary for them.

If L is the latus rectum of PP′, and DD′ its conjugate diameter,

$$PP' : DD' = DD' : L,$$
$$\therefore PP' : L = PP'^2 : DD'^2$$
$$= CP^2 : CD^2,$$

which is the form of the ratio which is generally employed at the present day.

In Props. 20, 21, after noticing that the fundamental properties of the parabola and central conic are equivalent to

$$QV^2 \propto PV,$$

and $$QV^2 : PV \cdot VP' = L : PP',$$

or as we now write it,

$$QV^2 : PV \cdot VP' = CD^2 : CP^2,$$

he proceeds to the consideration of a tangent, which he treats as a line which meets the curve in one point, but being produced both ways does not cut it. In this part of the subject the treatment is very similar to that employed in the third book of Euclid, *e.g. cf.* Euc. III. 16 with Ap. I. Prop. 32: "If through the vertex of a conic a straight line is drawn parallel to an ordinate, it touches the section, and no other straight line can be drawn between that straight line and the conic so as not to cut the curve."

In Prop. 34 his method of drawing a tangent at a point Q of a central conic (Fig. 2) is to draw the ordinate QV of any diameter CP, and take $CT \cdot CV = CP^2$, and show that TQ does not meet the curve at any other point but Q. Then the statement that this property holds in the case of the tangent follows as the converse of the above.

In Prop. 40 he shows that both in the ellipse and hyperbola the fundamental properties which were established for the primary diameter hold also for the secondary diameter, so that in Figs. 2 and 3,

$$Qu^2 : Cu \cdot ut = CP^2 : CD^2.$$

Hitherto he has not made mention of any diameter besides the primary and its conjugate, but in Prop. 46 and the following propositions he shows that any line is a diameter in the parabola if it is parallel to the primary diameter, and in the central conic if it passes through the centre; and in Props. 49, 50 he shows that the relations which he established between the ordinates, abscissæ, and latus rectum of the primary diameter are true for any diameter. He then concludes the first book with the description of (1) a conic, having given a diameter and its latus rectum, and (2) a hyperbola, having given a pair of conjugate diameters. Seeing that the latter datum is equivalent to having given either diameter with its latus rectum he obtains both a hyperbola and its conjugate.

Before leaving the first book it is as well to notice that in it Apollonius makes no mention of axis, focus, asymptote, or auxiliary circle, although we shall see that he was well acquainted with them, and that he opens the subject in a much more general manner than is now customary.

In Book II., which contains fifty-three propositions, he introduces asymptotes by taking points T, T′ on the tangent at P such that $TP^2 = T'P^2 = \frac{1}{4}PP' \times$ its latus rectum, and then showing that CT and CT′ if produced will never meet the curve, and are therefore called asymptotes.

He first proves the various properties relating to the segments of chords which intersect the asymptotes and one branch of the hyperbola, and it is not until Prop. 15 that he shows that the asymptotes are the same for both branches, and in 17 for the conjugate hyperbola, and he then shows that similar properties hold in their cases also. In 27 he considers the relations which exist between the diameter through the intersection of two tangents and their chord of contact, and having in 27-30 established them for any section of a cone, he shows in 31-40 that they hold for both branches of a hyperbola. He concludes the book with some problems such as: "Having given a conic, to find its centre and axes; to draw a tangent (1) from a given point, (2) making a given angle with the axis, or with the diameter through the point of contact, etc.

As before, the treatment is quite general, and it is only at the end of the book that we find mention made of the

axes, which are considered as diameters whose ordinates are at right angles to them.

The third book, containing fifty-six propositions, divides itself naturally into four parts.

It was proved in I. 43 (see Fig. 4) that the triangle QVF = the quadrilateral VPLK. From this proposition, which, as Eutocius tells us, has twelve particular cases for the parabola and hyperbola, and forty-two for the ellipse, Apollonius deduces three other properties which are little more than a statement of the same in another form, although, with his usual thoroughness, he employs the first fifteen propositions in proving them separately for each of the three sections. In 16-29 he then establishes the property that if two intersecting straight lines cut a conic, the rectangles contained by their segments are in the ratio of the squares on the parallel tangents. In 30-40 he lays the foundations of a part of the subject which in the hands of modern geometers has proved to be of the utmost importance. In Fig. 5 TP, TP′ are two tangents, PP′ their chord of contact, and TQEQ′ any chord through T. It is proved in 37 and 39 that TQEQ′ is a harmonic range; and if TF is drawn parallel to PP′ and through O, the mid-point of PP′, any chord R′ORF is drawn, 38 and 40 prove that R′ORF is a harmonic range. But PP′ is the polar of T, and TF is the polar of O. Therefore we have a complete proof of the property that "If any point be taken either within or without a conic, any chord through the point is divided harmonically by the curve, the point, and the polar of the point," which is the fundamental proposition in the theory of poles and polars. The particular cases in which the chord is parallel to an asymptote, or the point is on an asymptote, etc., had been previously considered in 30-36.

In 41 and two following propositions it is proved that in the parabola three tangents cut one another in the same ratio; if two parallel tangents to a central conic are intersected by a third tangent, the product of their segments is equal to the square on the parallel semi-diameter; and a tangent to a hyperbola makes with the asymptotes a triangle of constant area.

In 45 we are introduced for the first time to the foci of a central conic, although no special term is given to them except to indicate that they are points obtained by applying to the transverse axis a rectangle equal to the square on the semi-conjugate axis, exceeding or deficient by a square, according as the conic is a hyperbola or ellipse.* Here again we have the employment of Euc. VI. 28, 29.

He then proves, amongst other properties connected with the foci—

1. If the tangent at the vertices A, A′ are cut by any third tangent in Z, Z′ the intercept ZZ′ subtends a right angle at each focus.

2. A tangent makes equal angles with the focal distances of its point of contact.

3. The property of the pedal circle.

* τὰ ἐκ τῆς παραβολῆς γενόμενα σημεῖα.

4. The transverse axis is equal to the difference or sum of the focal distances of any point on the conic.

He then concludes the book with five propositions, in which he gives a full solution of the celebrated problem—the *locus ad tres lineas*—which he refers to in his general preface as having been attempted by Euclid, who was only able to solve it in a particular case. He states the problem in a different manner from that to which we are accustomed, but it is very easy to put it into the form with which we are familiar. The property which he proves may be briefly stated as follows:—

In Fig. 6 QE, Q′E′ are two given straight lines in which Q, Q′ are fixed points. Then if lengths QE, Q′E′ are taken such that the rectangle QE, Q′E′ is constant, and QE′, Q′E intersect in P, the locus of P is a conic passing through the points Q, Q′, and having its tangents at those points parallel to the given lines Q′E′, QE. Let us assume this result in the form in which Apollonius gives it in Prop. 54, viz.:—

$$QE \cdot Q'E' : QQ'^2 = \left\{ \begin{array}{l} RV^2 : TR^2 \\ TQ \cdot TQ' : QV \cdot VQ' \end{array} \right\}.$$

Draw PL, PM, PN perpendiculars on the sides of the triangle TQQ′. Draw PF, PF′ parallel to TQ, TQ′, and let C$a$, C$b$, C$c$ be the semi-diameters parallel to the given lines QE, Q′E′, QQ′ respectively.

Then noticing that RV : TR = R′V : R′T, we have from the similar triangles EQQ′, PF′Q′,

$$QE : QQ' = PF' : F'Q' \text{ (or } PK') = PN : PM,$$
$$Q'E' : QQ' = PF : FQ \text{ (or } PK) = PN : PL.$$

$$\therefore PN^2 : PL \cdot PM = QE \cdot Q'E' : QQ'^2,$$
$$= \left\{ \begin{array}{l} RV \cdot VR' : TR \cdot TR' \\ TQ \cdot TQ' : QV \cdot VQ' \end{array} \right\},$$
$$= \left\{ \begin{array}{l} CR^2 : Cc^2 \\ Ca \cdot Cb : CR^2 \end{array} \right\},$$
$$= Ca \cdot Cb : Cc^2,$$

which is of the form $\alpha\beta = K\gamma^2$, where $K = \dfrac{Cc^2}{Ca \cdot Cb}$.

From this we can easily deduce the *locus ad quatuor lineas*, in the form $\alpha\beta = K \cdot \gamma\delta$, in which $K = \dfrac{Cc \cdot Cd}{Ca \cdot Cb}$, where C$a$, C$b$, C$c$, C$d$ are the semi-diameters parallel to the sides of the quadrilateral, to which $\alpha$, $\beta$, $\gamma$, $\delta$ are respectively perpendicular.

Taking into consideration the distinct statement in his preface that in the third book he has given a complete investigation of the *locus ad tres et quatuor lineas*, I think it is quite possible that the form in which he gives it may have been the general statement of the problem in his day, and that by the time of Pappus, 600 years afterwards, it had assumed the form in which we now have it, but it is not easy to understand why the fact that Props. 52-56 of BK. III. contain a solution of this problem remained unnoticed for over 2000 years.

Book IV. contains fifty-five propositions, in the first eight of which he supposes one tangent to be drawn from an external point, and he obtains a second tangent by drawing any chord through the point, and dividing it

harmonically, and joining the point of section so found to the point of contact of the given tangent. This will intersect the curve in the point of contact of the required tangent. In the next fifteen propositions he shows how to draw two tangents from a point by drawing two chords through them and dividing them harmonically. The line joining the internal points of section gives the chord of contact of the required tangents.

The remaining thirty-two propositions deal with the number of points at the most in which two conics can intersect or touch one another without coinciding. As an example of the method employed in this book I will give briefly his proof of the proposition that two conics cannot intersect in more than four points.

In Fig. 7 suppose them to intersect in the four points A, B, C, D, which are taken in order so that there are no points of intersection lying between A and B, or B and C, or C and D. If possible, let E be a fifth point common to both conics. Let AB and DC meet in F, and in FBA, FCD take points G, H so that FBGA, FCHD are harmonic ranges. Then GH is the chord of contact of tangents from F to both conics. Therefore if EF be joined, meeting GH in K, and FKE be divided in L, so that FLKE is a harmonic range, the point L must lie on both conics, which is contrary to the supposition that there is no point of intersection between B and C.

Of the remaining four books, as I have said before, all traces of the Greek text have been lost, and we are dependent on a translation into Arabic, which was made by Thebit ben Corah at the beginning of the ninth century.

The fifth book, containing seventy-seven propositions, deals with normals, which are treated entirely as questions of maxima and minima. In the preface to it Apollonius says: "In dealing with this subject I had intended to keep to the same lines that I followed in the previous books of the elements of the three sections, that is, to investigate them generally with regard to any diameter; but as the number of properties is then very great, I have for the present confined my attention to that part of the subject which relates to the axes or principal diameters." The field which is here referred to is certainly a very wide and fertile one, and, as Apollonius says, "the subject belongs to the number of those which are worth investigating for their own sake."

With regard to this book we may remark that Euc. III. 7, 8 seem to have been taken as a model, and that the shortest line referred to is always the part of a normal intercepted between the curve and the transverse axis.

In Props. 1-15 it is shown that if we measure along the transverse axis a distance AG from the vertex equal to half the latus rectum of that axis, then this segment of the axis is for every point on it a minimum line to the conic; and that if AG be taken greater than the semi-latus rectum, then if we take CN : NG = AC : its semi-latus rectum, and draw the ordinate NP, then PG will be the shortest line which can be drawn from the point G to the conic; and of the rest, that which is nearer to it on either side of it is less than one more remote, and the difference between the squares on GP and on any other line GQ from G to the conic can be expressed as a certain rectangle described on the part of the axis intercepted between the ordinates of P and Q. In Props. 16-23 it is shown that similar properties hold with respect to the minor axis and its latus rectum. In Props. 27-36 we find that a minimum line is at right angles to the tangent at its extremity, and that if any number of normals be drawn, the acute angle made with the transverse axis by one which is nearer the vertex is less than that made by one more remote; and in Props. 38-48, if two shortest lines be drawn in the same quadrant, they will intersect in the opposite quadrant on the other side of the transverse axis; and from their point of intersection no other shortest line can be drawn to the first quadrant, and consequently not more than two normals can be drawn from any point to the same quadrant, nor more than three to the same semi-ellipse.

We now come to some important properties respecting normals which will be understood from Fig. 8.

C is the centre of an ellipse, O any point on the normal at P in the angle contained by CA and BC produced through C. OD, XOE are parallel to the axes, and K, L points on the axes such that

$$a^2 : b^2 = CK : KD,$$
$$= EL : LC,$$

and KX, LY are parallel to the axes.

Then it is proved in Prop. 45 that the rectangles $Pn.Pk$, OX.OY, CK.CL are all equal. This can be expressed geometrically in the form: If a rectangular hyperbola be described having KX and YL for asymptotes, and passing through the point O, it will pass through the centre and intersect the conic at the point where the normal through O meets it. In Prop. 52 we have the case where two of the points of intersection of this rectangular hyperbola with the given conic coincide, as at P, so that the point O, being the intersection of two consecutive normals, is a centre of curvature whose co-ordinates $\lambda$, $\mu$ are obtained in terms of those of P as follows:—

On the transverse axis CM and CN are two mean proportionals between CK and CA, so that

$$CK : CM = CM : CN = CN : CA,$$
$$\therefore CK : CN = CN^2 : a^2,$$

and by the previous construction

$$CD : CK = a^2 - b^2 : a^2,$$
$$\therefore CD : CN = \left\{ \begin{matrix} CN^2 : a^2 \\ a^2 - b^2 : a^2 \end{matrix} \right\},$$
$$\therefore \lambda \text{ or } CD = \frac{a^2 - b^2}{a^4} \cdot CN^3.$$

From this he obtains the value of $\mu$ or OD in terms of PN by taking

$$\mu : PN = \left\{ \begin{matrix} \lambda : DK \\ NK : NC \end{matrix} \right\} = \left\{ \begin{matrix} a^2 - b^2 : b^2 \\ PN^2 : b^2 \end{matrix} \right\},$$
$$\therefore \mu = \frac{a^2 - b^2}{b^4} \cdot PN^3.$$

In the demonstration of Prop. 52 it is shown that for any given value of AN greater than half the latus rectum of the transverse axis, the number of normals which can be drawn to the quadrant AB from any point Q in the perpendicular through D will be two, one, or none, according as the ordinate of the point Q is $<=>\mu$.

After considering particular positions of the point, he shows 55-63 that, in the general case, if the rectangular hyperbola referred to above be described, the branch which passes through the given point will always intersect the conic, and the straight line joining the given point to the point of intersection of the curves is a normal to the given conic. Since in general two conics intersect in four points, four normals can in general be drawn from a point to a conic.

If the ordinate of the point Q (Fig. 8) is $=$ or $>\mu$ in the cases of the parabola and hyperbola, or $>\mu$ in the case of the ellipse, then 64-67 QP is the shortest line from Q to the curve. Props. 68-71 show that of two tangents the one whose point of contact is nearer to a vertex of the transverse axis is less than the other. Prop. 73 proves in effect that the circle of curvature at any point P crosses the curve at P, the nearer vertex A being inside the circle. 74, 75 deal with the relative lengths of the different normals which can be drawn from any point, and 76, 77 are particular cases where the point is on the minor axis of an ellipse.

Book VI., containing thirty-three propositions, shows in what cases conics and their segments are similar, and when they are equal. Apollonius tells us that it contains other properties which had been passed over in silence by other writers; in particular, the method of cutting any right cone so as to obtain a section equal to a given section, and also how to construct a right cone which shall be similar to a given cone, and contain a given conic. "These things," he says, "I have treated more fully and clearly than previous writers on the subject."

He defines conics to be similar when, if corresponding axes are divided into an equal number of parts, or into parts which have the same ratio to another, the ratio of the ordinate to the abscissa at corresponding points is the same, and he thence deduces that all parabolas are similar, that two hyperbolas are similar if the ratio of the transverse axis to its latus rectum is the same in each, or if the ratio of any diameter to its latus rectum is the same in each, provided that the ordinates to those diameters make equal angles with them. The opposite branches of a hyperbola are equal and similar, and if a cone is cut by parallel planes, the sections are similar, but not equal. The book concludes with six propositions showing how to cut a required conic from a given cone, and to construct a cone similar to a given right cone, and containing a given conic.

The seventh book, containing fifty-one propositions, is almost entirely devoted to the consideration of properties relating to diameters and their latera recta. Amongst them it is proved in 12, 13 that in the hyperbola the difference, and in the ellipse the sum of the squares on any two conjugate diameters is equal to the difference or sum of the squares on the axes. In 23 any diameter of the equilateral hyperbola is equal to its conjugate. In 21-24 the ratio of the greater axis to the less is $>$ the ratio of any other diameter to its conjugate, and the difference between the axes is $>$ the difference between any two conjugate diameters, whilst the sum and product of the former are respectively $<$ the sum and product of the latter. In 31 the parallelogram whose sides are equal and parallel to a pair of conjugate diameters is equal to the rectangle contained by the axes.

I have added a detailed list of the properties proved in this book, so that they may be read at leisure.

In Fig. 9 Q is any point on the curve, and QN perpendicular to AA′. CP and CD are parallel to A′Q and AQ. On the axis take the point I such that

$$A'I : AI = AC^2 : BC^2,$$

and make A′I′ = AI. Let $2l$, $2l'$, 2L, 2L′ denote the latera recta of AA′, BB′, PP′, DD′ respectively. Then $PP' \cdot 2L = DD'^2$, *i.e.*

$$PP' : DD' = DD' : 2L,$$
$$\therefore PP' : 2L = PP'^2 : DD'^2$$
$$= CP^2 : CD^2.$$

Props.

2, 3. $A'Q^2 : A'N \cdot NI' = AA' : A'I'$.

6, 7. $CP^2 : CD^2 = NI' : NI$.

8. $AC^2 : (CP + CD)^2 = AI' \cdot NI : (NI' + \sqrt{NI : NI'})^2$.
$AC^2 : CD^2 = AI' : NI$.

9. $AC^2 : (CP - CD)^2 = AI \cdot I'N : (NI' - \sqrt{NI \cdot NI'})^2$.

10, 11. $AC^2 : CP \cdot CD = AI' : \sqrt{NI \cdot NI'}$.

12. $AC^2 + CB^2 = CP^2 + CD^2$ (ellipse).

13. $AC^2 - CB^2 = CP^2 - CD^2$ (hyperbola).

14. $AC^2 : CP^2 - CD^2 = AI' : 2CN$ (ellipse).

15. $AC^2 : L^2 = AI' \cdot NI : NI'^2$.

16. $AC^2 : (CP - L)^2 = AI' \cdot NI : (NI' - NI)^2$.

17, 19. $AC^2 : (CP + L)^2 = AI' \cdot NI : (NI' + NI)^2$.

18. $AC^2 : CP \cdot L = AI' : NI$. (*Cf.* 8.)

20. $AC^2 : CP^2 - L^2 = AI' \cdot NI : NI^2 - NI'^2$.

23. In the equilateral hyperbola conjugate diameters are equal to one another.

21-24. If AC>BC, and CP, CD are conjugates, of which CP>CD, then

$$AC : BC > CP : CD,$$

and if CP′ be a semi-diameter nearer to AC than CP is, and if CD′ be its conjugate, then

$$CP' : CD' > CP : CD.$$

25, 26. $AC + BC < CP + CD$.

27. $AC - BC > CP - CD$.

28. $AC \cdot BC < CP \cdot CD$.

29. $AC^2 - BC^2 = CP^2 - CD^2$ (hyperbola).

30. $AC^2 + BC^2 = CP^2 + CD^2$ (ellipse). (*Cf.* 12, 13.)

31. The parallelogram whose sides are equal and parallel to a pair of conjugate diameters is equal to the rectangle contained by the axes.

# THE CONICS OF APOLLONIUS.

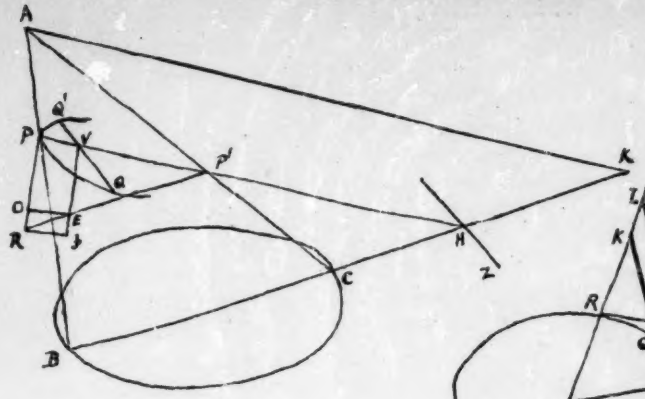


fig. 1.

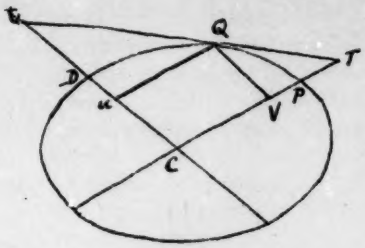


fig. 2.

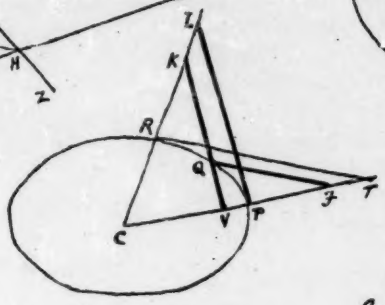


fig. 4

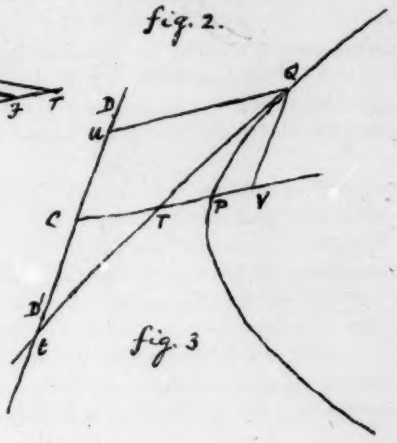


fig. 3

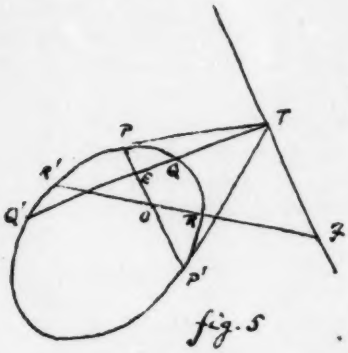


fig. 5

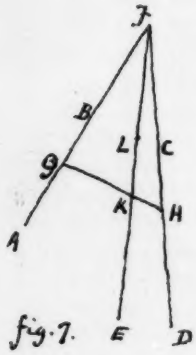


fig. 7.

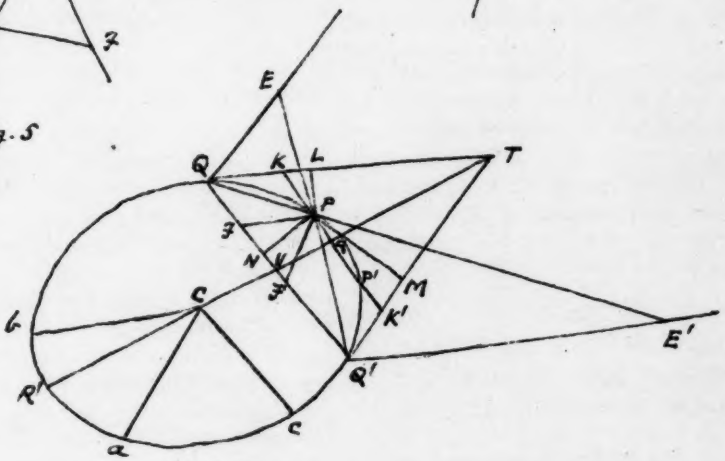


fig. 6

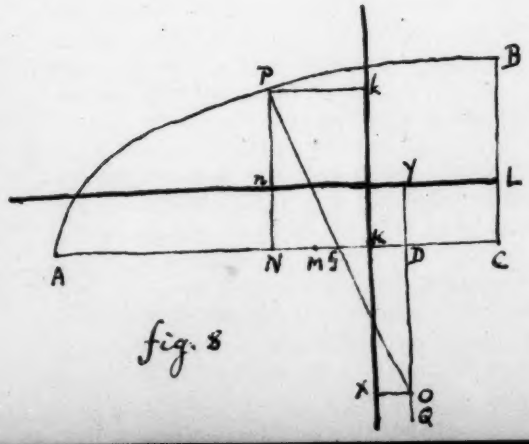


fig. 8

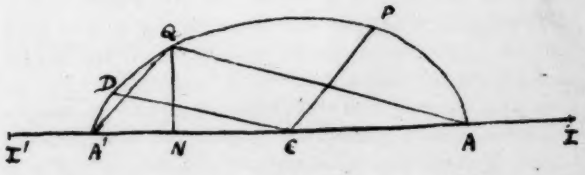


fig. 9.

32. In any parabola $l<L$.

33. In the hyperbola, if AC>BC, then $l<L$.

34. „ „ , if AC<$l$, $\not<\frac{1}{2}l$, then $l<L$.

35. „ „ , if AC<$\frac{1}{2}l$, then on each side of the axis there can be found a diameter whose latus rectum is double of the diameter, and this latus rectum is less than the latus rectum of any other transverse diameter.

36. In the hyperbola $AC - l>CP - L$.

37. In the ellipse $CB - l'>AC - l>CP - L$.

38, 39. In the hyperbola, if $AC \not< \frac{1}{3}l$, then $AC + l<CP + L$.

40. In the hyperbola, if AC<$\frac{1}{3}l$, then on each side of the axis there can be found a diameter whose latus rectum is three times the diameter, and the sum of this diameter and its latus rectum is a minimum.

41. In the ellipse $AC + l$ is a minimum, and $BC + l'$ is a maximum.

42, 43. $AC \cdot l$ is a minimum.

44, 45. In the hyperbola, if $AC \not< l$, or if AC<$l$, but $AC^2 \not< \frac{1}{2}(l - AC)^2$, then $AC^2 + l^2$ is a minimum.

46. In the hyperbola, if AC<$l$, and $AC^2<\frac{1}{2}(l - AC)^2$, then, as in 35, 40, two diameters can be found such that the square on each is equal to half the square on the difference between it and its latus rectum, and in this case $CP^2 + L^2$ is a minimum.

49. In the hyperbola, if AC>$l$, then $AC^2 - l^2$ is a minimum, and $2(AC^2 - AC \cdot l)>CP^2 - L^2>AC^2 - AC \cdot l$.

50. In the hyperbola, if AC<$l$, then $l^2 - AC^2$ is a maximum, and $CP^2 - L^2>2(AC^2 - AC \cdot l)$.

51. In the ellipse $AC^2 - l^2$ is a maximum for all diameters which are greater than their latera recta, whilst $l'^2 - BC^2$ is a maximum for all which are less.

With regard to the eighth book, which has been lost, Apollonius tells us that it contained a number of problems depending on the properties which had been proved in the seventh book, to which it was a sort of appendix. From this statement, together with the circumstance that Pappus gives the lemmas of the seventh and eighth books together, and not separately, as in the other books, Halley supplied its place with thirty-three propositions, in the first four of which a conic is supposed to be given, and also a diameter, and its latus rectum; and in the remaining twenty-nine there is given the axis and its latus rectum, but not the curve, and from these data various problems are solved, which depend to a great extent on the properties which were proved in the seventh book.

As we have seen, the ground covered by Apollonius is very extensive, and many parts of the subject are very thoroughly treated which are passed over in silence in modern text-books. Let us in conclusion consider what are the properties which are not to be found in the work of the great geometer.

1. We do not find any trace of the directrix of a conic.

2. Although we find the foci of a central conic, and some of the most important properties relating to them, there is no indication of the existence of a focus in the case of the parabola, nor any properties relating to focal chords, nor is there any reference to confocal conics.

3. We meet with several properties of the circle of curvature, although the circle itself is not mentioned, nor is the length of its radius, nor of the chord of curvature given.

4. With regard to normals, the relation

$$NG : NC = BC^2 : AC^2$$

is frequently used, but we do not find

$$CG \cdot CT = AC^2 \pm BC^2,$$

nor the general property of which the preceding is a particular case, viz., that if CP be a diameter bisecting any chord QQ', and if the tangents at Q, Q' intersect in T and the normals in O, and OR be perpendicular to CP, then

$$CR \cdot CT = CP^2 \pm CD^2.$$

5. Although it is proved that any straight line through a point is divided harmonically by the conic, the point, and its polar, we do not find any mention of either of the properties that if tangents are drawn from any point on the polar the chords of contact all pass through the pole, and that if any chord is drawn through the pole the tangents at its extremities intersect on the polar.

6. It is not stated that tangents are proportional to the diameters parallel to them, except in the particular case of conjugate diameters.

7. The only property given relating to the special case of the equilateral hyperbola is that if the axes are equal, then any diameter is equal to its conjugate; and although Apollonius has frequently to draw through a given point a hyperbola whose asymptotes are two given straight lines at right angles, he does not state that this hyperbola is equilateral.

JOHN J. MILNE.

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## PROOF OF HORNER'S METHOD OF APPROXIMATION TO A NUMERICAL ROOT OF AN EQUATION BY THE PROPERTIES OF ALGEBRAICAL QUOTIENTS AND REMAINDERS.

This short paper is supplementary to that read to the Association by Mr. Langley (Jan. 1892); that also being supplementary to one read by Mr. Hayward (Jan. 1889).

The elementary text-books on algebra now give the proof that if $f(x)$, an algebraical expression with integral indices, be divided by $x - a$ the remainder is $f(a)$; that is, we can find the remainder without actually performing the division by $x - a$ in the elementary way, but by taking the detached coefficients of $f(x)$, and putting them in the scale of $a$ by the rule in arithmetic called *reduction*. The gist of this rule, as shown in the most familiar example of turning pounds, shillings, and pence to pence, is that we do not turn the pounds to pence, the shillings to pence, and then add both of these to the remaining pence, but proceed by gradual accumulation in the well-known manner. In like manner here, except that the base of the scale is constant, viz. $a$, instead of changing from 20 to 12.

As the next step, the text-books on theory of equations show that the same process by which we find $f(a)$ gives us at the same time the coefficients of the integral quotient of $f(x)/(x-a)$, or using (A, B, C . . .) $a$ to denote A, B, C . . . put into the scale of $a$; $Q\{f(x)/(x-a)(x-b)(x-c)\ .\ .\ .\}$, $R\{f(x)/(x-a)(x-b)(x-c)\ .\ .\ .\}$, and $r\{f(x)/(x-a)(x-b)(x-c)\ .\ .\ .\}$ for the integral quotient the whole remainder, and the last partial remainder when $f(x)$ is divided compositely, first by $x-a$, then by $x-b$, and so on, they show, for every coefficient, if

$f(x) = Ax^n + Bx^{n-1} + \ldots + Hx^p + \ldots$, and the integral quotient $= A'x^{n-1} + B'x^{n-2} + H'x^{p-1} + \ldots$ that $H' = (A, B, C, \ldots H)a$.

But this is the same as showing that $r\{f(x)/(x-a)x^p\} = r\{f(x)/x^p(x-a)\}$.

Now this is true, because it is more generally true that $Q\{f(x)/(x-a)(x-b)(x-c)\ .\ .\ .\}$ and $R\{f(x)/(x-a)(x-b)(x-c)\ .\ .\ .\}$ are unaltered by any change in the order of the divisors; and the same is true for $r\{f(x)/(x-a)(x-b)(x-c)\ .\ .\ .\}$, because it is the coefficient of the highest power of $x$ in $R\{f(x)/(x-a)(x-b)(x-c)\ .\ .\ .\}$. To see that this constancy does not apply except with restrictions to the other partial remainders, take the case of four divisors, and let $r_1, r_2, r_3, r_4$ be the partial remainders, and

$$f(x) = f_4(x)\,.\,(x-a)(x-b)(x-c)(x-d) + r_4(x-a)(x-b)(x-c) + r_3(x-a)(x-b) + r_2(x-a) + r_1.$$

Here, from symmetry, $f_4(x)$ is independent of the order in which the four divisors are taken in the corresponding composite division; but $r_4$ is not merely independent of the order of division of the three factors $x-a$, $x-b$, and $x-c$, but is unaltered if $x-d$ takes the place of any of these three. For if

$$f(x) = f_4(x)\,.\,(x-d)(x-b)(x-c)(x-a) + \rho_4(x-d)(x-b)(x-c) + \rho_3(x-d)(x-b) + \rho_2(x-d) + \rho_1,$$

$\rho_4 = r_4$ for the reason above stated. In like manner $r_3$ would remain unaltered so long as $x-a$, $x-b$, $x-c$ were the first three divisors in any order.

It may be noted that in composite division in arithmetic if $N = N_4\,.\,abcd + r_4\,.\,abc + r_3\,.\,ab + r_2\,.\,a + r_1$, where the remainders $r_1, r_2, r_3, r_4$ are less than $a, b, c, d$ respectively, $r_1 + r_2.a$ is $< a + (b-1)a$ or $< ab$; and similarly $r_1 + r_2.a + r_3 ab < abc$, so that $r_4$ is the greatest integer in $R\{N/abcd\}/abc$, and therefore the last remainder of the four divisions of the corresponding composite division would be the same for any order of the first three divisors, but it would not in general be the same if the last divisor took the place of any of the preceding divisors, and in like manner in other cases.

Now let $f(a+b) = f(a) + b\,.\,f_1(a) + b^2\,.\,f_2(a) + b^3 f_3(a) + \ldots b^n\,.\,f_n(a)$.

This assumption is legitimate, because $f(a)$ being an algebraic function of the $n$th degree, the result of substituting $a+b$ for $a$ could be expressed in terms of ascending powers of $b$ as far as $b^n$, and the coefficients of those powers could be expressed as powers of $a$: also when these coefficients are found we have $f(a+b) = \{f_n(a), f_{n-1}(a)\ \ldots\ f_1(a), f(a)\}_b$; but it is not intended that $f_n(a), f_{n-1}(a)\ \ldots$, etc., should be less than $b$.

Let $a + b = X$, then

$$f(X) = f(a) + (X-a)\,.\,f_1(a) + (X-a)^2 f_2(a) + \ldots + (X-a)^n\,.\,f_n(a),$$

whence it follows that $f(a), f_1(a), f_2(a)\ \ldots\ f_n(a)$ are the successive partial remainders when $f(X)$ is divided $n$ times, compositely, by $X-a$.

Having formed a function of $b$, say $\phi(b)$, which is equal to $f(a+b)$, we may, by a repetition of the like process, that is, by a process equivalent to dividing $\phi(x)$ $n$ times by $x-b$, obtain from the coefficients of $\phi(x)$ fresh coefficients, which when put into the scale of $c$ give the value of $\phi(b+c)$, and therefore of $f(a+b+c)$, and so on.

The determination of the most appropriate values to give to $a$, $b$, $c$, etc., in the case of a given equation, is not peculiar to Horner's method, and need not here be described.

MORGAN JENKINS.

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## GREEK GEOMETERS BEFORE EUCLID.

(*Continued from No.* 3, *p.* 19.)

After Pythagoras and the Italic school, Professor Loria takes up the Eleatics, the Atomists, the Sophists. He expounds the arguments of Zeno (born about the beginning of the 5th century B.C.) against multiplicity and motion, and attributes to the Eleatic philosopher—he was not strictly a mathematician—the great merit of having for the first time discussed infinitesimal and infinite quantities as well as infinite numbers.

Oenopides of Chios flourished about 500 B.C., and is classed sometimes with the Pythagorean, sometimes with the Ionic school. He was an astronomer, but as a geometer his merit is somewhat doubtful. Proclus credits him with the solution of the two problems which constitute Euclid I. 12, 23. Oenopides called a perpendicular, after the archaic manner, a gnomon.

Anaxagoras of Clazomenæ was born about 500, and died about 428 B.C. Of his mathematical discoveries all we know is that, according to Plutarch, he wrote, while in prison, on the quadrature of the circle, and, according to Vitruvius, on perspective. His study of the latter subject was suggested by Agatharchus's painting of a scene for one of the tragedies of Aeschylus.

Democritus of Abdera was born about 460 B.C., and lived to a great age. His merit as a geometer is seen from the list of his writings. The title of one of them, "On the difference of the gnomon, or on the contact of the circle and the sphere" is, if not unintelligible, at least obscure; Allman's conjecture regarding it is quite different from Tannery's. He wrote also "On geometry," "On numbers," "On perspective," and two books on incommensurable lines and solids. The following remark of Democritus affords a glimpse of Egyptian geometry: "In

the construction of lines with demonstration no one has yet surpassed me, not even the so-called Harpedonaptae of Egypt." Dr. Moritz Cantor explains most ingeniously that these Harpedonaptae were rope-fasteners, and that in order to draw on the ground a line perpendicular to another, a rope divided into three parts proportional to 3, 4, 5 was stretched round three pegs. The operation of rope-stretching was one of great antiquity.

Proclus in his commentary on the first book of Euclid twice mentions a certain Hippias as the inventor of a curve called the Quadratrix, and historians are somewhat divided in opinion as to whether he is the same person as Hippias of Elis, the famous sophist, a contemporary of Socrates. Pappus, again, speaks of Deinostratus as the inventor of the Quadratrix. The explanation which Dr. Cantor gives and which Professor Loria accepts is that this curve was invented by Hippias for the trisection of an angle, that at first it had no name, but that those who afterwards applied it to the quadrature of the circle bestowed on it the name of Quadratrix.

Hippocrates of Chios flourished in the second half of the 5th century B.C. His contributions to geometry are of two kinds,—indirect, that is methodological, and direct. He was the first writer of a book of *Elements* of geometry, and he is said to have invented the method of reduction, that is, the method by which one problem or theorem is reduced to another problem or theorem, the resolution of the latter involving that of the former. It seems difficult to admit that this natural and spontaneous method of geometrical research had not been made use of by geometers anterior to Hippocrates; probably the merit of Hippocrates consisted in exposing the logical substratum common to many methods of solution previously used, and in applying it to new examples. His direct contributions concerned the two famous problems,—the duplication of the cube and the quadrature of the circle. The first of these problems, which is one in solid geometry, Hippocrates transformed into another in plane geometry, namely the insertion of two mean proportionals between two given straight lines. In his attempt to solve the second problem Hippocrates shows how certain lunules could be squared, and in the course of his demonstration makes use explicitly or implicitly of a considerable number of theorems and problems, several of which are to be found in Euclid's *Elements*.

Antiphon attempted to measure the circle by inscribing in it a series of regular polygons, the number of whose sides form a geometrical progression with common ratio 2; Bryson made the same attempt by inscribing and circumscribing regular polygons. These methods of obtaining the quadrature of the circle were condemned by some of the ancient geometers, but they are substantially the method of exhaustions.

J. S. MACKAY.

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## NOTES.

20. On some suggestions in mathematical terminology.

(i.) To designate the line which bisects at right angles the join of two points A, B, the term *axis of A, B* is proposed. Reasons: (1) brevity; (2) avoidance of the suggestion that the line joining AB is necessary in constructing it (important in teaching geometrical drawing); (3) two points, like any other pair of circles, have a *radical axis* which is the line in question.

(ii.) To designate the tangent of the angle of inclination to the horizontal, or to the $x$-axis, or to any straight line or direction of reference: *gradient* is proposed. Reasons: (1) present engineering usage; (2) as compared with the word *slope* it has a more definite suggestion of the way it is to be measured, *i.e.*, by trigonometrical tangent, not by angular magnitude; (3) already in use by several authorities.

(iii.) For unit of *moment of a force* on the British system of units: *pound-foot* and *poundal-foot* for the gravitational and absolute unit respectively.

(iv.) For inverse trigonometrical and hyperbolic functions, *nis, soc, nat, toc, ces, cesoc, nish, soch, nath, toch, cesh, cesoch*, in place of $sin^{-1}$, $cos^{-1}$, etc., and of the continental *arc sin, arc sin hyp*, etc. R. F. MUIRHEAD.

21. On some trigonometrical identities.

(i.) From the point O where the internal bisector of A meets the circumcircle of triangle ABC, draw OP, OQ perpendicular to CA, AB.

Then
$$\begin{aligned} \angle AOP &= \tfrac{1}{2}\angle POQ \\ &= \tfrac{1}{2}(B + C), \\ \angle COP &= \angle COA - \angle AOP \\ &= B - \tfrac{1}{2}(B + C) \\ &= \tfrac{1}{2}(B - C). \end{aligned}$$

In triangles OPC, OQB
$$\begin{aligned} OC &= OB \\ OP &= OQ \\ \text{rt. } \angle OPC &= \text{rt. } \angle OQB, \\ \therefore PC &= BQ \\ \therefore AP + AQ &= b + c \\ \therefore AP &= \tfrac{1}{2}(b + c) \\ \therefore PC &= \tfrac{1}{2}(b - c) \\ \therefore \tfrac{1}{2}(b - c) &= OP \tan \tfrac{1}{2}(B - C) \end{aligned}$$

and
$$\begin{aligned} \tfrac{1}{2}(b + c) &= OP \tan \tfrac{1}{2}(B + C), \\ \therefore \frac{b - c}{b + c} &= \frac{\tan \tfrac{1}{2}(B - C)}{\tan \tfrac{1}{2}(B + C)}. \end{aligned}$$

Let PQ cut BC in D and AO in H.

Then PH is at right angles to AO, and it may easily be shown (Simpson line demonstration) that OD is perpendicular to BC, and therefore bisects it.

$$\therefore \frac{HP}{OP} = \frac{DC}{OC}.$$

But $$OP = OC \cos \tfrac{1}{2}(B - C),$$

$$\therefore HP = \frac{a}{2} \cos \tfrac{1}{2}(B - C),$$

$$\therefore \tfrac{1}{2}(b + c) \sin \frac{A}{2} = \frac{a}{2} \cos \tfrac{1}{2}(B - C).$$

Also $$\frac{\tfrac{1}{2}(b - c)}{\sin \tfrac{1}{2}(B - C)} = OC$$

$$= \frac{\tfrac{1}{2}a}{\sin \tfrac{1}{2}(B + C)}.$$

The figure is of wide and remarkable use in dealing with the various conic sections, and the trigonometrical use of it was suggested by one of Dr. Taylor's demonstrations. I hope to make it the subject of another note.

E. M. LANGLEY.

(ii.) *Proof.*—$\tan \overline{A + B} = \dfrac{\tan A + \tan B}{1 - \tan A \,.\, \tan B}$.

Draw angles A, B above and below FG.

Make FG = 1, draw HGK perpendicular to it,

$\therefore$ HG = tan A, GK = tan B.

Draw HLM perpendicular to FK,

$\therefore$ M = G = rt. $\angle$, $\therefore$ FMGH concyclic,

$\therefore$ LHG = B, $\therefore$ LG = tan A . tan B,

$\therefore$ FL = 1 − tan A . tan B.

Also right-angled triangles FML, HMK similar,

$\therefore$ MH/FM = HK/FL,

$\therefore \tan \overline{A + B} = (\tan A + \tan B)/(1 - \tan A \,.\, \tan B)$.

J. H. HOOKER.

22. On Simpson's Rule.

I wish to call attention to the importance of Simpson's Rule in finding the volume of a solid. All the simple solids, such as the sphere, cone, pyramid, wedge, paraboloid of revolution, and their frusta, and also spheroids and ellipsoids, obey the rule accurately.

The rule is: If a solid is bounded by two parallel planes, the average cross-section parallel to these planes is obtained (either accurately or approximately) by adding together the two end sections and four times the parallel section mid-way between them, and dividing by 6. The volume is this average section multiplied by the distance between the end sections.

Take, for example, the sphere. The end sections are both zero, and the mid-section = $\pi r^2$, therefore the average section = $\frac{1}{6}(0 + 0 + 4\pi r^2) = \frac{2}{3}\pi r^2$. Multiply this by $2r$, which is the height of the solid, and we obtain the volume $\frac{4}{3}\pi r^3$.

Similarly in all the other cases.

The volumes of all prismoids are found by the rule.

If the solid is an irregular one, as in the case of earthwork, the rule gives a good approximate value for the average section. In case of great irregularity, the solid must be divided up into several lengths by parallel planes, and the rule applied to each portion.

Another point is the importance of teaching students to verify their formulæ by testing their accuracy in simple cases.

For example, the average cross-section of a frustum of a circular cone, whether right or oblique, is $\frac{\pi}{3}(a^2 + ab + b^2)$, where $a$, $b$ are the radii of the two circular ends. This is obtained at once by Simpson's rule. It can be tested by applying it (1) to a whole cone, (2) to a thin slice, or, which is the same thing, by letting the frustum become a cylinder.

Again, the mid-section of a spherical frustum is

$$\tfrac{1}{2}(\pi a^2 + \pi b^2) + \tfrac{1}{4}\pi h^2.$$

The accuracy of this can be tested by taking (1) a whole sphere, (2) a thin slice. If desired also, a third test would be by taking a symmetrical frustum.

Similarly, the average section of a spherical frustum, viz.,

$$\tfrac{1}{2}(\pi a^2 + \pi b^2) + \tfrac{1}{6}\pi h^2$$

can be tested by the tests (1) and (2), and also by taking a hemisphere, whose average section is necessarily equal to that of a whole sphere.

In fact, the values of the coefficients in the expression

$$A(a^2 + b^2) + Bab + Ch^2$$

can be found, according as the expression is to represent the average section of a conical or pyramidal frustum, or of a spherical frustum, or the mid-section of either, by applying the above tests.

A third important point is that the students should, as far as possible, find the volumes of actual solids, by handling them and making the necessary measurements. In fact, mensuration should be taught, not only as a branch of mathematics but as an experimental science. In these days there is perhaps the less need to dwell on the importance of this point, as teachers are in all departments recognising the importance of experiments conducted by the pupils themselves. At the same time I wonder how many boys who know, for instance, that the area of a rectangle is obtained by multiplying its length and breadth, have actually measured the length and breadth of any rectangle, such as a table or desk lid, and then calculated its area. And perhaps still fewer have been set to work to find the volume of a solid. A cask is a very good example to which to apply Simpson's Rule.

In connection with the gauging of casks, the following rules are useful.

To find the number of gallons of water in an upright cylinder, multiply the depth of the water in yards by the square of the diameter of the cylinder in inches, and divide by 10. The result is the number of gallons to an accuracy of 2 per cent.

Further:—

If the operation is performed without dividing by 10, the result is the weight of water in lbs., since a gallon of water weighs 10 lbs.

*Proof of 1st theorem*:—Let depth = $h$ yards, and diameter = $d$ inches.

Then quantity of water in the cylinder

$$= \frac{\pi}{4}\left(\frac{d}{12}\right)^2 (3h) \text{ cubic feet}$$

$$= \frac{\pi}{192}(d^2h) \text{ cubic feet}$$

$$= \frac{\pi}{192} \times 6{\cdot}23(d^2h) \text{ gallons,}$$

which is about equal to $\frac{1}{10}\left(1+\frac{1}{50}\right)(d^2h)$ gallons,

$\therefore$ quantity of water $= \frac{1}{10}\left(1+\frac{1}{50}\right)d^2h$ gallons,

and weight of water $= \left(1+\frac{1}{50}\right)d^2h$ lbs.

If the vessel is not a cylinder, the value of $d^2$ for the equivalent cylinder is given by Simpson's Rule, viz.

$$d^2 = \frac{1}{6}\left(d_1^2 + 4d_2^2 + d_3^2\right),$$

if $d_1$, $d_3$ are the end diameters, and $d_2$ the mid-diameter (measured in inches) of the vessel.

I hope in a subsequent number to give some applications of Simpson's Rule to plane areas.

ALFRED LODGE.

23. On (i.) division into classes, and (ii.) homogeneous products.

The formation of homogeneous products may be connected with the division of a number of given things into classes by the following method, in which dividing lines are regarded as things to be permutated with the rest.

(1) To find the number of ways of dividing $n$ like things into $r$ different classes, blank classes being admissible.

Place the $n$ things in a line and then divide into $r$ classes by $r-1$ dividing lines. These lines may be placed in intervals between consecutive things, any number in one interval, or one or more of them may be placed at either end of the row. Thus, representing like things by dots, any order of eight dots and five dividing lines, as the following,

| | · · · · | · | | · · ·

indicates one way of dividing eight like things into six different classes, the numbers in the classes in this case being respectively 0, 0, 4, 1, 0, 3.

In the general case, therefore, each possible division into classes is indicated by an order amongst the $n$ like things and the $r-1$ like dividing lines; and hence the number required is the same as the number of permutations of $n+r-1$ things of which $n$ are alike and also the remaining $r-1$ alike: that is,

$$\frac{\lfloor n+r-1}{\lfloor n \,\lfloor r-1}$$

(2) If blank classes are inadmissible, or more generally if there must be at least $q$ in each class, then, after placing $q$ in each class, this method can be applied to the remaining $n-rq$ things that must now be divided amongst the $r$ classes. The number of ways is therefore

$$\frac{\lfloor n-rq+r-1}{\lfloor n-rq \;\lfloor r-1}$$

(3) If some or all of the $n$ things are different the method of section (1) can be used, but it must be noted that the result takes into account the order in each class.

Thus the number of ways of arranging $n$ different things in $r$ different classes is

$$\frac{\lfloor n+r-1}{\lfloor r-1}$$

and the number of ways of arranging a a a b b c in three classes is the same as the number of permutations of a a a b b c | |, viz.

$$\frac{\lfloor 8}{\lfloor 3 \,\lfloor 2 \,\lfloor 2}$$

(4) The number of homogeneous products of $r$ dimensions formed from $n$ quantities is the same as the number of combinations that can be formed from $n$ things when there are $r$ things in each group, but repetitions *ad libitum* are allowed. Whichever view be taken, since each group is only counted once, whatever may be the order of its component things, we may suppose that there is a definite order according to which the things forming each group are to be arranged, or in other words a recognised order of precedence amongst the $n$ given things. This being agreed upon, we may next suppose that we form any one of the groups required by taking a row of $r$ vacant places and then filling each vacant place by one of the $n$ things, repetitions being allowed. The order of precedence being followed in filling the vacant places, it will be sufficient to mark off by a dividing line those places into each of which the first thing is to be put, then by a second dividing line those places into each of which the second thing is to be put, and so on. The number of dividing lines required will be $n-1$. Thus if we have to form groups of eight letters from the six a b c d e f, allowing repetitions, and if the order of precedence is alphabetical, then representing vacant places by dots, any order of eight dots and five lines indicates one of the groups required: *e.g.*, taking the homogeneous product view in this case,

· · | · | · | · · | · indicates $a^2$ b c d $e^2$ f,
· · · | | | | · · · · · indicates $a^3$ $f^5$,
| | · · | | · · · · · · · | indicates $c^2$ $e^6$,
· · · · · · · · | | | | | indicates $a^8$.

In the general case therefore each possible group is indicated by an order amongst the $r$ dots and the $n-1$ dividing lines; and hence the number of groups or of homogeneous products is

$$\frac{\lfloor n+r-1}{\lfloor r \;\lfloor n-1}$$

PERCY J. HARDING.

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## QUESTIONS FOR SOLUTION BY STUDENTS.

58. Construct the points whose pedal lines with respect to a given triangle pass through a given point, and show that there are in general three solutions.

W. C. FLETCHER.

59. Express the function $\Sigma yz$ as the sum of three positive or three negative squares. W. J. GREENSTREET.

60. To inscribe three mutually tangent circles in an isosceles triangle. PROF. D. S. WRIGHT.

61. In the trapezoid ABCD the parallel sides AB, CD are constants, and meet the variable base BD at right angles: prove that the perpendicular to BD from the intersection of the diagonals AD, BC is constant.

PROF. D. S. WRIGHT.

## REVIEWS AND NOTICES.

*A Treatise on Computation.* By E. M. LANGLEY. (Longmans, 1895.) Great attention has of late years been paid to quick methods of performing numerical calculations, and not a few papers and leaflets been written on the subject, but these have of necessity reached but a limited circle of readers.

The present book presents in a systematic form the latest results of these endeavours, and should be welcomed by all good teachers of arithmetic. It is divided into four sections, the first being devoted to the best methods of performing arithmetical operations (including Horner's method of finding cube roots), and the second to abbreviated means of obtaining approximate results, attention being called to the fact that in scientific calculations the data are only approximately accurate, and that no calculated results can be more accurate than the data. The third section is on logarithmic work, and is worthy of most careful attention by every student and teacher. The fourth section works out some practical examples, illustrating the methods of the book. It is a book that cannot be read in junior classes, but should be in the hands of every teacher, who should use its methods in such classes, and it should be worked through by the students as soon as they are sufficiently advanced.

The book hardly deals at all with mercantile computation, except so far as that is decimal.

The following two misprints have been noticed.

P. 7, Ex. (5), 1985673 should be two separate numbers, viz. 1985 and 673.

P. 183, No. 32. The fraction $\frac{1}{100}$ should be $\frac{1}{1400}$.

The following addition might be made to No. 28, on p. 182, as giving an exceedingly accurate series for reducing feet to metres:—

$$\frac{3}{10}+\frac{1}{200}-\frac{1}{5000}=\cdot 3048.$$

A. LODGE.

*Geometrical Conics.* By F. S. MACAULAY, M.A. (Cambridge University Press.) After a short chapter on the forms of the parabola, ellipse, and hyperbola, the author devotes three chapters to a separate treatment of these curves, and proceeds to a consideration of the properties of the conic generally. Poles and Polars have a chapter to themselves, as also have the Rectangular Hyperbola and the right Circular Cone. Some interesting miscellaneous theorems from Chasles, Salmon, Graves, etc., are given, and an introductory treatment given of Cross Ratio, Imaginary Points, and the Principle of Continuity. Unfortunately, the book did not arrive in time to be reviewed in company with those on the same subject by Smith, Milne and Davis.

I. *Elementary Trigonometry.* By C. PENDLEBURY, M.A. (G. Bell & Sons.)

II. *The Elements of Trigonometry.* By A. S. GHOSH. (Patrick Press, Calcutta.) These works have almost the same scope, and are marked by the clearness of exposition and the attention to students' difficulties which might be expected from the experience of the authors, both as writers and teachers. Each is a serviceable treatise, and in each judicious use is made of typographical distinctions. Mr. Pendlebury gives special attention to the solution of equations and to the theory of projection, and adds to the value of his book by the insertion of a list of the formulæ which a student should by degrees render part of his mental armoury. As in his *Arithmetic*, the value of oral examples is insisted on. Mr. Ghosh's book is a worthy companion to his editions of his father's *Elements of Geometry* and *Arithmetic for Colleges and Schools*, reviewed in our last issue. We wish that both authors could have seen their way to the disestablishment of the tabular function and its "big L." We are strongly of De Morgan's opinion in this matter.

*Arithmetic Prize Papers.* By W. P. WORKMAN, M.A., B.Sc. (Hughes & Co.) A set of papers by which the annual Kingswood School silver medal "to the best Arithmetician" has been awarded. The objects of the compiler have been to evoke reasoning power more than mechanism, to test knowledge of principles rather than of rules, and to make his papers "as hard as it is possible to make Arithmetic." He seems to have thoroughly succeeded in these objects, especially in the last, and has humanely added full solutions. The book should be in the hands of every mathematical master.

*The Elements of Co-ordinate Geometry.* By S. L. LONEY, M.A. (Macmillan & Co.) An excellent treatise for junior students; very thorough within the limits fixed by the author. We can cordially recommend it.

Algebra. Part I. By M. H. SENIOR. (D. W. Bardsley, Oldham.) Price 4d.

*Geometry Tablet for written Exercises.* By BEMAN and SMITH. (Ginn and Co.) An exercise book with upward of sixty ruled pages, with margin for teachers' criticisms, to be expressed briefly by means of a code explained on the cover.

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*Journal de Mathématiques Élémentaires.* June-Oct. 1895.

*Periodico di Matematica.* May-Aug. 1895.